Please check the examination details below before entering your candidate information

Candidate surname	Other names

Centre Number | Candidate Number

Pearson Edexcel International Advanced Level

Time 1 hour 30 minutes

Paper reference **WFM01/01**

Mathematics

International Advanced Subsidiary/ Advanced Level
Further Pure Mathematics F1

You must have:
Mathematical Formulae and Statistics Tables (Yellow), calculator

Total Marks

Candidates may use any calculator allowed by Pearson regulations. Calculators must not have the facility for symbolic algebra manipulation, differentiation and integration, or have retrievable mathematical formulae stored in them.

Instructions

- Use **black** ink or ball-point pen.
- If pencil is used for diagrams/sketches/graphs it must be dark (HB or B).
- **Fill in the boxes** at the top of this page with your name, centre number and candidate number.
- Answer **all** questions and ensure that your answers to parts of questions are clearly labelled.
- Answer the questions in the spaces provided – *there may be more space than you need.*
- You should show sufficient working to make your methods clear. Answers without working may not gain full credit.
- Inexact answers should be given to three significant figures unless otherwise stated.

Information

- A booklet 'Mathematical Formulae and Statistical Tables' is provided.
- There are 9 questions in this question paper. The total mark for this paper is 75.
- The marks for **each** question are shown in brackets – *use this as a guide as to how much time to spend on each question.*

Advice

- Read each question carefully before you start to answer it.
- Try to answer every question.
- Check your answers if you have time at the end.
- If you change your mind about an answer, cross it out and put your new answer and any working underneath.

Turn over

P 7 2 4 6 4 A 0 1 3 2

Pearson

1. Given that

$$\mathbf{A} = \begin{pmatrix} 2 & -1 & 3 \\ -2 & 3 & 0 \end{pmatrix} \quad \text{and} \quad \mathbf{B} = \begin{pmatrix} 1 & k \\ 0 & -3 \\ 2k & 2 \end{pmatrix}$$

where k is a non-zero constant,

(a) determine the matrix **AB**

(2)

(b) determine the value of k for which det(**AB**) = 0

(3)

a. matrix A : 2×3

matrix B : 3×2

$A \times B = (2 \times 3) \times (3 \times 2)$ this is the new size of the matrix (2×2)

these 2 no.s are the same so can do $A \times B$

$$\begin{bmatrix} 2 & -1 & 3 \\ -2 & 3 & 0 \end{bmatrix} \times \begin{bmatrix} 1 & k \\ 0 & -3 \\ 2k & 2 \end{bmatrix} = \begin{bmatrix} (2)(1)+(-1)(0)+(3)(2k) & (2)(k)+(-1)(-3)+(3)(2) \\ (-2)(1)+(3)(0)+(0)(2k) & (-2)(k)+(3)(-3)+(0)(2) \end{bmatrix}$$

$$= \begin{bmatrix} 2+6k & 2k+9 \\ -2 & -2k-9 \end{bmatrix}$$

$$\boxed{AB = \begin{bmatrix} 2+6k & 2k+9 \\ -2 & -2k-9 \end{bmatrix}}$$

b. $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$\det(x) = (a)(d) - (b)(c)$

$$AB = \begin{bmatrix} 2+6k & 2k+9 \\ -2 & -2k-9 \end{bmatrix}$$

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Question 1 continued

$\det(AB) = (2+6k)(-2k-9) - (-2)(2k+9)$

$-\cancel{4}k - \cancel{18} - 12k^2 - 54k + \cancel{4}k + \cancel{18}$

$-12k^2 - 54$

$\det(AB) = -12k^2 - 54$

Since $\det(AB) = 0 \Rightarrow -12k^2 - 54 = 0$ $\times -1$

$12k^2 + 54k = 0$ factorise k out

$k(12k + 54) = 0$

$k = 0$ or $k = -\frac{54}{12}$

However, Q states k is a non-zero constant. $\therefore k = -\frac{54}{12} = -\frac{9}{2}$

$\boxed{k = \frac{-9}{2}}$

(Total for Question 1 is 5 marks)

2. **In this question you must show all stages of your working.**

Solutions relying entirely on calculator technology are not acceptable.

Use the standard results for $\sum_{r=1}^{n} r$ and $\sum_{r=1}^{n} r^2$ to show that for all positive integers n

$$\sum_{r=1}^{n}(7r-5)^2 = \frac{n}{6}(7n+1)(An+B)$$

where A and B are integers to be determined.

(6)

$$\sum_{r=1}^{n}(7r-5)^2 = \sum_{r=1}^{n} 49r^2 - 70r + 25$$

Split Summation into 3

$$\sum_{r=1}^{n} 49r^2 - \sum_{r=1}^{n} 70r + \sum_{r=1}^{n} 25$$

factor 49 out summation

factor 70 out summation

factor 25 out summation

$$49\sum_{r=1}^{n} r^2 - 70\sum_{r=1}^{n} r + 25\sum_{r=1}^{n} 1$$

Summations

$\sum_{r=1}^{n} 1 = n$

$\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$

must learn!

$\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$

$\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2$

in the formulae booklet

$$49\left[\frac{1}{6}n(n+1)(n+2)\right] - 70\left[\frac{1}{2}n(n+1)\right] + 25[n]$$

$$\frac{49}{6}n(n+1)(n+2) - \frac{70}{2}n(n+1) + 25n$$

factor $\frac{1}{6}n$ out

$$\frac{1}{6}n\left[49(n+1)(n+2) - 210(n+1) + 150\right]$$

$$\frac{1}{6}n\left[49(2n^2+3n+1) - 210(n+1) + 150\right]$$

$$\frac{1}{6}n\left[98n^2 + 147n + 49 - 210n - 210 + 150\right]$$

$$\frac{1}{6}n\left[98n^2 - 63n - 11\right]$$

$$\boxed{\frac{n}{6}(7n+1)(14n-11) \quad A = 14,\ B = -11}$$

3. **In this question you must show all stages of your working.**

Solutions relying entirely on calculator technology are not acceptable.

$$f(z) = 4z^3 + pz^2 - 24z + 108$$

where p is a constant.

Given that -3 is a root of the equation $f(z) = 0$

(a) determine the value of p **(2)**

(b) using algebra, solve $f(z) = 0$ completely, giving the roots in simplest form, **(4)**

(c) determine the modulus of the complex roots of $f(z) = 0$ **(2)**

(d) show the roots of $f(z) = 0$ on a single Argand diagram. **(2)**

a. Since -3 is a root of the eq^n, $f(-3) = 0$

$f(-3) = 4(-3)^3 + p(-3)^2 - 24(-3) + 108 = 0$

$-\cancel{108} + 9p + 72 + \cancel{108} = 0$

$9p + 72 = 0$

$9p = -72$

$p = \frac{-72}{9} = -8$

$\boxed{p = -8}$

b. For a cubic eq^n with form: $ax^3 + bx^2 + cx + d$, there are 3 roots.

There are 2 possibilities: 3 real roots

1 real root and 2 complex roots (conjugate pair)

$f(z) = (z+3)(az^2 + bz + c)$

$= az^3 + bz^2 + cz + 3az^2 + 3bz + 3c$

$= (a)z^3 + (b+3a)z^2 + (c+3b)z + (3c)$

Question 3 continued

Comparing coefficents with eq^n above : (1) $a = 4$

(2) $b + 3a = -8$

(3) $c + 3b = -24$

(4) $3c = 108$

(1) $a = 4$, (4) $3c = 108$

$c = \frac{108}{3}$

$c = 36$

Solve for b by subbing a into (2).

$b + 3a = -8$

$b + 3(4) = -8$

$b + 12 = -8$

$b = -20$

$\therefore$ quadratic root is $4z^2 - 20z + 36$

$4z^2 - 20z + 36 = 0$ $\div 4$

$z^2 - 5z + 9 = 0$

$\left(z - \frac{5}{2}\right)^2 - \frac{25}{4} + 9 = 0$

$\left(z - \frac{5}{2}\right)^2 + \frac{11}{4} = 0$

$\left(z - \frac{5}{2}\right)^2 = -\frac{11}{4}$

$\left(z - \frac{5}{2}\right) = \pm\sqrt{\frac{-11}{4}}$

$z = \frac{5}{2} = \pm\sqrt{\frac{11}{4}}\,i$

$z = \frac{5}{2} \pm \frac{\sqrt{11}}{2}\,i$

Solve for z
via completing the square

$\boxed{\text{Complex roots : } \frac{5 \pm \sqrt{11}\,i}{2}}$

Question 3 continued

c. Modulus of either complex root will give same answer, as they are complex conjugates.

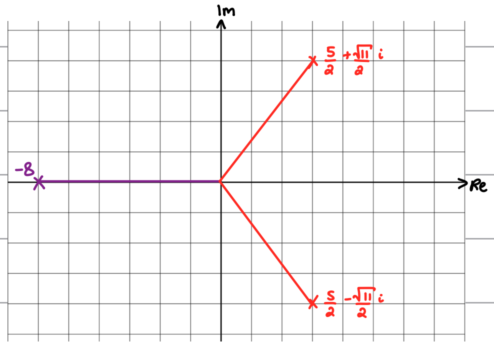
$\left| \frac{5}{2} + \frac{\sqrt{11}}{2} i \right|$

$= \sqrt{\left(\frac{5}{2}\right)^2 + \left(\frac{\sqrt{11}}{2}\right)^2}$

$= 3$

$\boxed{3}$

d.



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4.

$$f(x) = 1 - \frac{1}{8x^4} + \frac{2}{7\sqrt{x^7}} \qquad x > 0$$

The equation f(x) = 0 has a single root, α, that lies in the interval $[0.15, 0.25]$

(a) (i) Determine f′(x)

(ii) Explain why 0.25 cannot be used as an initial approximation for α in the Newton-Raphson process.

(iii) Taking 0.15 as a first approximation to α apply the Newton-Raphson process once to f(x) to obtain a second approximation to α
Give your answer to 3 decimal places.

(5)

(b) Use linear interpolation once on the interval $[0.15, 0.25]$ to find another approximation to α
Give your answer to 3 decimal places.

(3)

ai. Rewrite f(x) as indices:

$f(x) = 1 - \frac{1}{8}x^{-4} + \frac{2}{7}x^{-7/2}$

$f'(x) = \frac{1}{2}x^{-5} - x^{-9/2}$

$\boxed{f'(x) = \frac{1}{2}x^{-5} - x^{-9/2}}$

ii. Numerical Sol^n s of Eq^n s:

Newton-Raphson iteration for solving $f(x)=0$

$X_{n+1} = X_n - \frac{f(x_n)}{f'(x_n)}$

$f'(0.25) = \frac{1}{2}(0.25)^{-5} - (0.25)^{-9/2} = 0$

However, in formulae would mean $X_1 = 0.25 - \frac{f(0.25)}{0}$

cannot divide by 0.

$\boxed{\therefore \text{ 0.25 cannot be used as initial approximation.}}$

P 7 2 4 6 4 A 0 1 0 3 2

Question 4 continued

iii. $X_1 = 0.15$

we are looking for X_2

Numerical Sol^{n}s of Eq^{n}s:

Newton-Rhapson iteration for solving $f(x)=0$

$$X_{n+1} = X_n - \frac{f(X_n)}{f'(X_n)}$$

From Newton-Rhapson iteration: $X_2 = X_1 - \frac{f(X_1)}{f'(X_1)}$

(1) must find $f(X_1)$ by subbing in X_1 (0.15) into $f(x)$

(2) must find $f'(X_1)$ by differentiating $f(x)$ and subbing in X_1 (0.15) into $f'(x)$.

(1) $f(0.15) = 1 - \frac{1}{8(0.15)^4} + \frac{2}{\sqrt[7]{(0.15)^7}} = -27.33250956$

(2) $f'(x) = \frac{1}{2}x^{-5} - x^{-9/2}$ from part (ai)

$f'(0.15) = \frac{1}{2}(0.15)^{-5} - (0.15)^{-9/2} = 1484.137157$

$X_2 = (0.15) - \frac{-27.33250956}{1484.137157}$

$X_2 = 0.168416431$

$\boxed{\alpha = 0.168 \text{ (3d.p.)}}$

Question 4 continued

d. Linear Interpolation:

$$\alpha = \frac{af(b) - bf(a)}{f(b) - f(a)}$$ for interval $[a,b]$

where α is the root

$a = 0.15$, $b = 0.25$

$$\alpha = \frac{0.15\, f(0.25) - 0.25\, f(0.15)}{f(0.25) - f(0.15)}$$

$f(0.15) = -27.33250956$ from part (aiii)

$$f(0.25) = 1 - \frac{1}{8(0.25)^4} + \frac{2}{7\sqrt{(0.25)^7}} = \frac{39}{7}$$

$$\alpha = \frac{0.15\left(\frac{39}{7}\right) - 0.25(-27.33250956)}{\frac{39}{7} - -27.33250956}$$

$\alpha = 0.2330675935$

$\boxed{\alpha = 0.233 \ (3d.p.)}$

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5. The quadratic equation

$$4x^2 + 3x + k = 0$$

where k is an integer, has roots α and β

(a) Write down, in terms of k where appropriate, the value of $\alpha + \beta$ and the value of $\alpha\beta$

(2)

(b) Determine, in simplest form in terms of k, the value of $\dfrac{\alpha}{\beta^2} + \dfrac{\beta}{\alpha^2}$

(4)

(c) Determine a quadratic equation which has roots

$$\frac{\alpha}{\beta^2} \text{ and } \frac{\beta}{\alpha^2}$$

giving your answer in the form $px^2 + qx + r = 0$ where p, q and r are integer values in terms of k

(3)

a. $x^2 - (\text{sum of roots})x + (\text{product of roots}) = 0$

let α, β be roots of quadratic

$x^2 - (\alpha + \beta)x + (\alpha\beta) = 0$

sum of roots : $\alpha + \beta$

product of roots: $\alpha\beta$

$4x^2 + 3x + k = 0$

$x^2 + \frac{3}{4}x + \frac{k}{4} = 0$

$\alpha + \beta = -\frac{3}{4}$

$\alpha\beta = \frac{k}{4}$

$\boxed{\alpha + \beta = -\frac{3}{4} \quad \alpha\beta = \frac{k}{4}}$

b. $\dfrac{\alpha}{\beta^2} + \dfrac{\beta}{\alpha^2} = \dfrac{\alpha(\alpha^2) + \beta(\beta^2)}{\alpha^2\beta^2} = \dfrac{\alpha^3 + \beta^3}{(\alpha\beta)^2}$

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Question 5 continued

$$= \frac{(\alpha+\beta)^3 - 3\alpha\beta(\alpha+\beta)}{(\alpha\beta)^2}$$

$$= \frac{\left(-\frac{3}{4}\right)^3 - 3\left(\frac{k}{4}\right)\left(-\frac{3}{4}\right)}{\left(\frac{k}{4}\right)^2}$$

$$= \frac{-\frac{27}{64} + \frac{9k}{16}}{\frac{k^2}{16}} = \frac{\frac{-27+4(9k)}{64}}{\frac{k^2}{16}} = \frac{\frac{-27+36k}{64}}{\frac{k^2}{16}}$$

$$\frac{-27+36k}{64} \div \frac{k^2}{16}$$

$$= \frac{-27+36k}{\cancel{64}\,4} \times \frac{\cancel{16}\,1}{k^2}$$

$$= \frac{-27+36k}{4k^2}$$

$$\boxed{\frac{\alpha}{\beta^2} + \frac{\beta}{\alpha^2} = \frac{36k-27}{4k^2}}$$

c. $x^2 - (\text{sum of roots})x + (\text{product of roots}) = 0$

$$x^2 - \left(\frac{\alpha}{\beta^2} + \frac{\beta}{\alpha^2}\right)x + \left(\frac{\alpha}{\beta^2} \times \frac{\beta}{\alpha^2}\right) = 0$$

can use part (b) ans.

work out: $\frac{\cancel{\alpha}}{\beta^{\cancel{2}}} \times \frac{\cancel{\beta}}{\alpha^{\cancel{2}}} = \frac{1}{\alpha\beta} = (\alpha\beta)^{-1}$

$= \left(\frac{k}{4}\right)^{-1} = \frac{4}{k}$ //

$$x^2 - \left(\frac{36k-27}{4k^2}\right)x + \left(\frac{4}{k}\right) = 0$$

$\times 4k^2$

$$4kx^2 - (36k-27)x + 16k = 0$$

all integer coefficients

$$\boxed{4k^2x^2 - (36k-27)x + 16k = 0 \quad p = 4k^2,\ q = -(36-27k),\ r = 16k}$$

6. **In this question you must show all stages of your working.**

Solutions relying entirely on calculator technology are not acceptable.

The rectangular hyperbola H has equation $xy = 20$

The point $P\left(2t\sqrt{a}, \dfrac{2\sqrt{a}}{t}\right)$, $t \neq 0$, where a is a constant, is a general point on H

(a) State the value of a

(1)

(b) Show that the normal to H at the point P has equation

$$ty - t^3x - 2\sqrt{5}\left(1 - t^4\right) = 0$$

(4)

The points A and B lie on H

The point A has parameter $t = c$ and the point B has parameter $t = -\dfrac{1}{2c}$, where c is a constant.

The normal to H at A meets H again at B

(c) Determine the possible values of c

(4)

a. sub $x = 2t\sqrt{a}$, $y = \frac{2\sqrt{a}}{t}$ into $xy = 20$

$2\cancel{t}\sqrt{a} \times \frac{2\sqrt{a}}{\cancel{t}} = 20$

$2\sqrt{a} \times 2\sqrt{a} = 20$

$4a = 20$

$a = 5$

$\boxed{a = 5}$

b. Differentiate rectangular hyperbola eqⁿ w.r.t. x:

$xy = 20$

$y = \frac{20}{x} = 20x^{-1}$

$\frac{dy}{dx} = -20x^{-2} = \frac{-20}{x^2}$

Question 6 continued

sub in point $P\left(2t\sqrt{5}, \frac{2\sqrt{5}}{t}\right)$

$$\left.\frac{dy}{dx}\right|_{\left(2t\sqrt{5}, \frac{2\sqrt{5}}{t}\right)} \quad \frac{-20}{(2t\sqrt{5})^2} = \frac{-20}{20t^2} = \frac{-1}{t^2}$$

$M_{tangent} : \frac{-1}{t^2}$

$M_{normal} : t^2$

set up line eq^n $y - y_1 = m(x - x_1)$ with $m : t^2$

$(x_1, y_1) = \left(2t\sqrt{5}, \frac{2\sqrt{5}}{t}\right)$

$$y - \frac{2\sqrt{5}}{t} = t^2\left(x - 2t\sqrt{5}\right)$$

$$y - \frac{2\sqrt{5}}{t} = t^2 x - 2t^3\sqrt{5}$$

$$t\left(y - \frac{2\sqrt{5}}{t}\right) = t\left(t^2 x - 2t^3\sqrt{5}\right)$$

$$ty - 2\sqrt{5} = t^3 x - 2t^4\sqrt{5}$$

$$ty - t^3 x - 2\sqrt{5} + 2t^4\sqrt{5} = 0$$

$$\boxed{ty - t^3 x - 2\sqrt{5}(1 - t^4) = 0 \;// \quad \text{shown.}}$$

c. find point A:

sub $t = c$ into $P\left(2t\sqrt{5}, \frac{2\sqrt{5}}{t}\right)$

$A : \left(2c\sqrt{5}, \frac{2\sqrt{5}}{c}\right)$

find point B:

sub $t = \frac{-1}{2c}$ into $P\left(2t\sqrt{5}, \frac{2\sqrt{5}}{t}\right)$

$B : \left(2\left(-\frac{1}{2c}\right)\sqrt{5}, \frac{2\sqrt{5}}{-1/2c}\right) = \left(-\frac{\sqrt{5}}{c}, -4c\sqrt{5}\right)$

since normal eq^n crosses through A and B, sub in x & y coord. of A and B into normal eq^n.

Question 6 continued

subbing in $B\left(\frac{-\sqrt{5}}{c}, -4c\sqrt{5}\right)$

normal eq^n: $ty - t^3x - 2\sqrt{5}(1-t^4) = 0$

$: cy - c^3x - 2\sqrt{5}(1-c^4) = 0$

$: c(-4c\sqrt{5}) - c^3\left(\frac{-\sqrt{5}}{c}\right) - 2\sqrt{5}(1-c^4) = 0$

$: -4c^2\sqrt{5} + \sqrt{5}c^2 - 2\sqrt{5} + 2\sqrt{5}c^4 = 0$

$: 2\sqrt{5}c^4 - 3\sqrt{5}c^2 - 2\sqrt{5} = 0$ $\div\sqrt{5}$

$: 2c^4 - 3c^2 - 2 = 0$

$: (2c^2+1)(c^2-2) = 0$

$c^2 = -\frac{1}{2}$ or $c^2 = 2$

$\times$ $\quad$ $c = \pm\sqrt{2}$

$\boxed{c = \pm\sqrt{2}}$

7. (i)

$$\mathbf{P} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

The matrix **P** represents a geometrical transformation U

(a) Describe U fully as a single geometrical transformation.

(2)

The transformation V, represented by the 2×2 matrix **Q**, is a rotation through 240° anticlockwise about the origin followed by an enlargement about (0, 0) with scale factor 6

(b) Determine the matrix **Q**, giving each entry in exact numerical form.

(2)

Given that U followed by V is the transformation T, which is represented by the matrix **R**

(c) determine the matrix **R**

(2)

(ii) The transformation W is represented by the matrix

$$\begin{pmatrix} -2 & 2\sqrt{3} \\ 2\sqrt{3} & 2 \end{pmatrix}$$

Show that there is a real number λ for which W maps the point $(\lambda, 1)$ onto the point $(4\lambda, 4)$, giving the exact value of λ

(5)

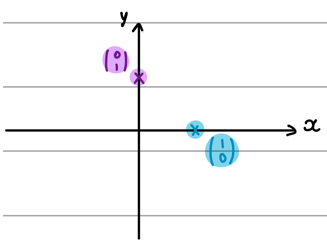
a. Start by drawing a coordinate axis and mark coordinates $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

then mark the coordinates of new matrix

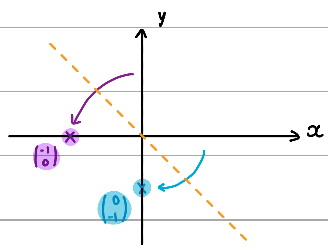
identity matrix I

matrix P

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow{\text{transformation}} \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$



transformation represents reflection against line $y = -x$



$\boxed{\text{Reflection in the line } y = -x}$

Question 7 continued

b.

$\begin{bmatrix} \cos\vartheta & -\sin\vartheta \\ \sin\vartheta & \cos\vartheta \end{bmatrix}$ ← Anticlockwise rotation through ϑ about O (Given in Formulae booklet)

$\vartheta = 240° \longrightarrow \begin{bmatrix} \cos(240°) & -\sin(240°) \\ \sin(240°) & \cos(240°) \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$

$6 \times \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} -3 & 3\sqrt{3} \\ -3\sqrt{3} & -3 \end{bmatrix}$

scale factor 6

$\boxed{Q : \begin{bmatrix} -3 & 3\sqrt{3} \\ -3\sqrt{3} & -3 \end{bmatrix}}$

c. order : $R = QP$

$R = \begin{bmatrix} -3 & 3\sqrt{3} \\ -3\sqrt{3} & -3 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

$R = \begin{bmatrix} (-3)(0) + (3\sqrt{3})(-1) & (-3)(-1) + (3\sqrt{3})(0) \\ (-3\sqrt{3})(0) + (-3)(-1) & (-3\sqrt{3})(-1) + (-3)(0) \end{bmatrix}$

$\boxed{R = \begin{bmatrix} -3\sqrt{3} & 3 \\ 3 & 3\sqrt{3} \end{bmatrix}}$

ii. $\begin{bmatrix} -2 & 2\sqrt{3} \\ 2\sqrt{3} & 2 \end{bmatrix} \begin{bmatrix} \lambda \\ 1 \end{bmatrix} = \begin{bmatrix} 4\lambda \\ 4 \end{bmatrix}$

$\begin{bmatrix} (-2)(\lambda) + (2\sqrt{3})(1) \\ (2\sqrt{3})(\lambda) + (2)(1) \end{bmatrix} = \begin{bmatrix} 4\lambda \\ 4 \end{bmatrix}$

Question 7 continued

$-2\lambda + 2\sqrt{3} = 4\lambda$

$2\sqrt{3} = 6\lambda$

$\lambda = \frac{\sqrt{3}}{3}$

$2\sqrt{3}\lambda + 2 = 4$

$2\sqrt{3}\lambda = 2$

$\lambda = \frac{2}{2\sqrt{3}} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$

$\boxed{\lambda = \frac{\sqrt{3}}{3}}$

8. A parabola C has equation $y^2 = 4ax$ where a is a positive constant.

The point S is the focus of C

The line l_1 with equation $y = k$ where k is a positive constant, intersects C at the point P

(a) Show that

$$PS = \frac{k^2 + 4a^2}{4a}$$

(3)

The line l_2 passes through P and intersects the directrix of C on the x-axis.

The line l_2 intersects the y-axis at the point A

(b) Show that the y coordinate of A is $\dfrac{4a^2k}{k^2 + 4a^2}$

(3)

The line l_1 intersects the directrix of C at the point B

Given that the areas of triangles BPA and OSP, where O is the origin, satisfy the ratio

area BPA : area OSP = $4k^2 : 1$

(c) determine the exact value of a

(5)

a. $y^2 = 4ax$

focus: $(a, 0)$

$\therefore$ point S $(a, 0)$

Conics

	Ellipse	Parabola	Hyperbola	Rectangular Hyperbola
Standard Form	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$	$y^2 = 4ax$	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	$xy = c^2$
Parametric Form	$(a\cos\theta, b\sin\theta)$	$(at^2, 2at)$	$(a\sec\theta, b\tan\theta)$ $(\pm a\cosh\theta, b\sinh\theta)$	$\left(ct, \frac{c}{t}\right)$
Eccentricity	$e < 1$ $b^2 = a^2(1 - e^2)$	$e = 1$	$e > 1$ $b^2 = a^2(e^2 - 1)$	$e = \sqrt{2}$
Foci	$(\pm ae, 0)$	$(a, 0)$	$(\pm ae, 0)$	$(\pm\sqrt{2}c, \pm\sqrt{2}c)$
Directrices	$x = \pm\frac{a}{e}$	$x = -a$	$x = \pm\frac{a}{e}$	$x + y = \pm\sqrt{2}c$
Asymptotes	none	none	$\frac{x}{a} = \pm\frac{y}{b}$	$x = 0, y = 0$

Find point P by solving $y = k$ and $y^2 = 4ax$ simultaneously.

(1) $y = k$

(2) $y^2 = 4ax$

$y = k$

$y^2 = k^2$

square both sides

sub in $y^2 = k^2$ into (2)

$k^2 = 4ax$

$x = \frac{k^2}{4a}$

$\therefore P : \left(\frac{k^2}{4a}, k\right)$

Question 8 continued

Distance between 2 points: $\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$ for (x_1,y_1) and (x_2,y_2)

$$\|PS\| = \sqrt{\left(\frac{k^2}{4a}-a\right)^2+(k-0)^2} \qquad (a,0),\left(\frac{k^2}{4a^2},k\right)$$

$$= \sqrt{\left(\frac{k^2}{4a}-a\right)^2+(k)^2}$$

$$= \sqrt{\left(\frac{k^2-4a^2}{4a}\right)^2+k^2}$$

$$= \sqrt{\frac{(k^2-4a^2)^2}{(4a)^2}+k^2}$$

$$= \sqrt{\frac{(k^2-4a^2)^2+16a^2k^2}{(4a)^2}}$$

$$= \sqrt{\frac{k^4-8a^2k^2+16a^4+16a^2k^2}{(4a)^2}}$$

$$= \sqrt{\frac{k^4+8a^2k^2+16a^4}{(4a)^2}}$$

$$= \sqrt{\frac{(k^2+4a^2)^2}{(4a)^2}}$$

$$= \frac{\cancel{\sqrt{}}(k^2+4a^2)^{\cancel{2}}}{\cancel{\sqrt{}}(4a)^{\cancel{2}}}$$

$$= \frac{k^2+4a^2}{4a}$$

$$\boxed{\therefore \|PS\| = \frac{k^2+4a^2}{4a} \text{ // shown}}$$

b. ℓ_2 passes through $\underline{\underline{P}}$ and intersects directrix of C on x-axis.

$\left(\frac{k^2}{4a},k\right)$

directrix : $x=-a$

on x-axis, set $y=0$

$\therefore (-a,0) \leftarrow$ point where directrix crosses x-axis.

$$M_{\ell_2} = \frac{k-0}{\frac{k^2}{4a}--a} = \frac{k}{\frac{k^2}{4a}+a} = \frac{k}{\frac{k^2+4a^2}{4a}} = \frac{4ak}{k^2+4a^2}$$

Question 8 continued

Set up line eq^n $y - y_1 = m(x - x_1)$ with $m = \frac{4ak}{k^2+4a^2}$

$(x_1, y_1) = (-a, 0)$

$y - 0 = \frac{4ak}{k^2+4a^2}(x - -a)$

$y = \frac{4ak}{k^2+4a^2}(x + a)$

$l_2 : y = \frac{4ak}{k^2+4a^2}x + \frac{4a^2k}{k^2+4a^2}$

@ y-axis, $x = 0$ $\therefore$ set $x = 0$

$y = \frac{4ak}{k^2+4a^2}(0) + \frac{4a^2k}{k^2+4a^2}$

$y = \frac{4a^2k}{k^2+4a^2}$

$\boxed{\therefore \text{ y-coord. of A is } \frac{4a^2k}{k^2+4a^2} \text{ // shown}}$

C. $\Delta BPA : \Delta OSA = 4k^2 : 1$

$\frac{\Delta BPA}{\Delta OSA} = \frac{4k^2}{1} \Leftrightarrow \Delta BPA = 4k^2 \, \Delta OSP$

l_1 intersects directrix of C @ point B

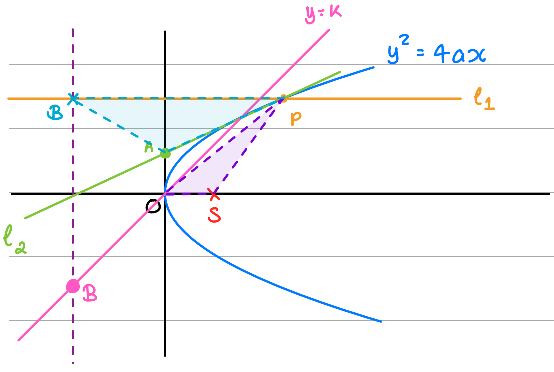
$\therefore$ solve l_1 and directrix line simultaneously.

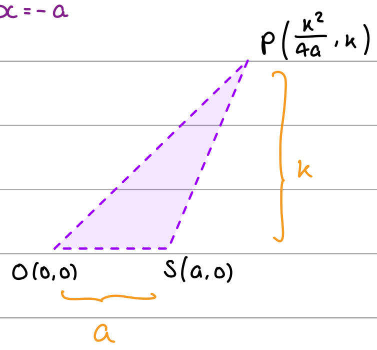
(1) $y = k$

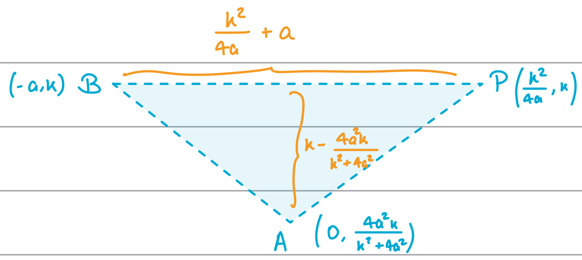
(2) $x = -a$

$\therefore B(-a, k)$

Question 8 continued







$\Delta OSP = \frac{1}{2} \times a \times k$

$\Delta OSP = \frac{ak}{2}$

$\Delta BPA = \frac{1}{2} \times \left(\frac{k^2}{4a} + a\right) \times \left(k - \frac{4a^2k}{k^2+4a^2}\right)$

$= \frac{1}{2} \times \left(\frac{k^2+4a^2}{4a}\right) \times \left(\frac{k^3+4a^2k-4a^2k}{k^2+4a^2}\right)$

$\Delta BPA = 4k^2 \ \Delta OSP$

$\frac{1}{2} \times \left(\frac{k^2+4a^2}{4a}\right) \times \left(\frac{k^3+4a^2k-4a^2k}{k^2+4a^2}\right) = 4k^2 \times \frac{ak}{2}$

$\frac{1}{2} \times \left(\frac{k^2+4a^2}{4a}\right) \times \left(\frac{k^3+4a^2k-4a^2k}{k^2+4a^2}\right) = 2ak^3$ $\times 2$

$\left(\frac{\cancel{k^2+4a^2}}{4a}\right) \times \left(\frac{k^3+4a^2k-4a^2k}{\cancel{k^2+4a^2}}\right) = 4ak^3$

$\frac{k^3+4a^2k-4a^2k}{4a} = 4ak^3$

$\cancel{k}(k^2+4a^2-4a^2) = 16a^2k^{\cancel{3}\,2}$

$\cancel{k^2} = 16a^2\cancel{k^2}$

$1 = 16a^2$

$a^2 = \frac{1}{16}$

$a = \pm\sqrt{\frac{1}{16}} = \pm\frac{1}{4}$

Q states, however, a is a positive constant $\therefore a = \frac{1}{4}$ only.

$\boxed{a = \frac{1}{4}}$

(Total for Question 8 is 11 marks)

9. Prove by induction that for all positive integers n

$$\sum_{r=1}^{n} \log(2r-1) = \log\left(\frac{(2n)!}{2^n n!}\right)$$

(6)

i. (1) Base Case: Prove true for $n=1$

LHS: $\sum_{r=1}^{1} \log(2r-1) = \log(2(1)-1) = 0$

RHS: $\log\left(\frac{(2(1))!}{2^1 \times 1!}\right) = \log\left(\frac{2!}{2\times 1!}\right) = 0$

LHS = RHS

$\therefore$ true for $n=1$

(2) Assume true for $n=k$:

$\sum_{r=1}^{k} \log(2r-1) = \log\left(\frac{(2k)!}{2^k k!}\right)$

(3) Prove true for $n=k+1$:

$\sum_{r=1}^{k+1} \log(2r-1) = \sum_{r=1}^{k} \log(2r-1) + \log(2(k+1)-1) \leftarrow$ we are rewriting sum of (k+1) terms as the sum of (k) terms + $(k+1)^{th}$ term – we can now use step (2).

Thinking ahead, sub in $n=k+1$ into RHS: this is what we are trying to prove using (2):

$\sum_{r=1}^{k+1} \log(2r-1) = \log\left(\frac{(2(k+1))!}{2^{k+1}(k+1)!}\right)$

└ Make a note so we know what we are working towards

$= \log\left(\frac{(2k)!}{2^k k!}\right) + \log(2k+1)$

$\log_a(m) + \log_a(n) = \log_a(m \times n)$

$= \log\left(\frac{(2k+1)\times(2k)!}{2^k k!}\right)$

$= \log\left(\frac{(2k+1)!}{2^k k!}\right)$

$= \log\left(\frac{(2k+1)!}{2^k k!} \times \frac{(2k+2)}{(2k+2)}\right)$

$= \log\left(\frac{(2k+2)(2k+1)!}{2^k k! \times 2(k+1)}\right)$

$= \log\left(\frac{(2k+2)!}{2^{k+1}(k+1)!}\right) = \log\left(\frac{(2k+1+1)}{2^{k+1}(k+1)!}\right)$ $\therefore$ true for $n=k+1$

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Question 9 continued

(4) Conclusion

If this result is true for $n=k$, then it is true for $n=k+1$. As the result has been shown to be true for $n=1$, then the result is true for all n.